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A New Filled Function Method for Global Optimization

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Abstract. A novel filled function is suggested in this paper for identifying a global minimum point for a general class of nonlinear programming problems with a closed bounded domain. Theoretical and numerical properties of the proposed filled function are investigated and a solution algorithm is proposed. The implementation of the algorithm on several test problems is reported with satisfactory numerical results.

Key words: Mathematical programming, Global optimization, Nonconvex optimization, Filled function method

1. Introduction

Optimization, as a powerful solution approach, finds wide applications in almost all fields of engineering, finance, management as well as social science. The existence of multiple local minima of a general nonconvex objective function makes global optimization a great challenge (see, e.g., [4, 7, 13]). The literature on global optimization can be classified into three categories. The first category includes methods that search for a global minimum among the local minima, more specifically, methods that invoke certain auxiliary functions to move successively from one local minimum to another better one (see, e.g., [1, 2, 5, 6, 9, 18]). The second category includes methods that use heuristic or stochastic search (see, e.g., [3, 15]). The third category includes methods that confine their applicability to problems with special structures, such as indefinite quadratic programming, concave minimization and D.C. programming (see, e.g., [7, 8, 14]). Recent work in [10] reveals that via certain convexification, concavification and monotonization schemes a nonconvex optimization problem with box constraints or over a simplex can be always converted into an equivalent better-structured nonconvex optimization problem, e.g., a concave optimization problem or a D.C. programming problem, thus facilitating the search of a global optimum by using the existing methods in concave minimization and D.C. programming. The results in [10] is further extended in [16].

This paper considers the following global optimization problem:

$$(P) \qquad \min_{x \in X} f(x)$$

where $X \subset \mathbb{R}^n$ is a closed bounded domain containing all global minimizers of $f(x)$ in its interior. It is assumed in this paper that $f(x)$ has only a finite number of local minimizers.

When $f : \mathbb{R}^n \to \mathbb{R}$ is coercive, i.e., $f(x) \to +\infty$ as $\|x\| \to +\infty$, then a global optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x)$$

can be always reduced into an equivalent problem formulation in (P).

The basin of $f(x)$ at an isolated minimizer of $f(x)$, x_1^*, is defined in [4, 5] as a connected domain B_1^* which contains x_1^* and in which the steepest descent trajectory of $f(x)$ converges to x_1^* from any initial point. The minimal radius of B_1^* at an isolated minimizer x_1^* is defined as

$$R = \inf_{x \notin B_1^*} \|x - x_1^*\|$$

Radius R is not zero if the Hessian at x_1^*, $\nabla^2 f(x_1^*)$, is positive definite. The basin of $f(x)$ at x_1^* is said to be lower than another basin of $f(x)$ at x_0^* if and only if $f(x_1^*) < f(x_0^*)$. The hill of $f(x)$ at $\hat{x}_1$ is the basin of $-f(x)$ at its isolated minimizer $\hat{x}_1$.

The concept of the filled functions was introduced by Ge in [5]. Assume that x_1^* is a local minimizer of $f(x)$. A function $p(x, x_1^*)$ is said to be a filled function of $f(x)$ at the local minimizer x_1^* if it satisfies the following:

(P1) x_1^* is a maximizer of $p(x, x_1^*)$ and the whole basin B_1^* of $f(x)$ at x_1^* becomes a part of a hill of $p(x, x_1^*)$;

(P2) $p(x, x_1^*)$ has no minimizers or saddle points in any basin of $f(x)$ higher than B_1^*;

(P3) if $f(x)$ has a basin B_2^* at x_2^* that is lower than B_1^*, then there is a point $x' \in B_2^*$ that minimizes $p(x, x_1^*)$ on the line through x_1^* and x'', for every x'' in some neighborhoods of x_2^*.

Note that (P3) in our paper is a revised version of the property (3) in [5]. Property (P3) in this paper is much stronger since a minimizer is required for lines connecting the current minimum with every point in some neighborhoods of a next better minimum, not just for one such point as required in property (3) in [5].

Adopting the concept of the filled functions, a global optimization problem can be solved via a two-phase cycle. In Phase 1, we start from a given point and use any local minimization method to find a local minimizer x_1^* of $f(x)$. In Phase 2, we construct a filled function at x_1^* and minimize the filled function in order to identify a point x' with $f(x') < f(x_1^*)$. If such a x' is found, x' is certainly in a lower basin than B_1^*. We can then use x' as the initial point in Phase 1 again, and hence we can find a better minimizer x_2^* of $f(x)$ with $f(x_2^*) < f(x_1^*)$. This process repeats until the time when minimizing a filled function does not yield a better solution. The current local minimum will be then taken as a global minimizer of $f(x)$.

Ge specifically proposed the following two-parameter filled function in [5]:

$$P(x, x_1^*, r, \rho) = \frac{1}{r + f(x)} \exp\left(-\frac{\|x - x_1^*\|^2}{\rho^2}\right), \tag{1.1}$$

where the parameters r and ρ need to be chosen appropriately. Ge and Qin in [6] noticed an unfavorable property in numerical implementation of the filled function in (1.1). Since both $P(x, x_1^*, r, \rho)$ and $\nabla P(x, x_1^*, r, \rho)$ are affected by the term $\exp(-\|x - x_1^*\|^2/\rho^2)$, changes in $P(x, x_1^*, r, \rho)$ and $\nabla P(x, x_1^*, r, \rho)$ become indistinguishable when $\|x - x_1^*\|$ is large. Ge and Qin in [6] tried to overcome this problem by proposing seven other filled functions: $\tilde{P}(x, x_1^*, r, \rho)$, $G(x, x_1^*, r, \rho)$, $\tilde{G}(x, x_1^*, r, \rho)$, $Q(x, x_1^*, A)$, $\tilde{Q}(x, x_1^*, A)$, $E(x, x_1^*, A)$ and $\tilde{E}(x, x_1^*, A)$, where

$$\tilde{P}(x, x_1^*, r, \rho) = \frac{1}{r + f(x)} \exp\left(-\frac{\|x - x_1^*\|}{\rho^2}\right), \tag{1.2}$$

$$G(x, x_1^*, r, \rho) = -\rho^2 \log[r + f(x)] - \|x - x_1^*\|^2, \tag{1.3}$$

$$\tilde{G}(x, x_1^*, r, \rho) = -\rho^2 \log[r + f(x)] - \|x - x_1^*\|, \tag{1.4}$$

$$Q(x, x_1^*, A) = -[f(x) - f(x_1^*)] \exp(A\|x - x_1^*\|^2), \tag{1.5}$$

$$\tilde{Q}(x, x_1^*, A) = -[f(x) - f(x_1^*)] \exp(A\|x - x_1^*\|), \tag{1.6}$$

$$\nabla E(x, x_1^*, A) = -\nabla f(x) - 2A[f(x) - f(x_1^*)](x - x_1^*), \tag{1.7}$$

$$\nabla \tilde{E}(x, x_1^*, A) = -\nabla f(x) - A[f(x) - f(x_1^*)] \frac{x - x_1^*}{\|x - x_1^*\|}. \tag{1.8}$$

They pointed out that the last four are better choices for filled functions. Nevertheless, the expressions of $E(x, x_1^*, A)$ and $\tilde{E}(x, x_1^*, A)$ are unknown. It is also evident that filled functions in forms of (1.5) and (1.6) still suffer the same handicap as (1.1). Recently, Liu [11] proposed a new filled function to tackle this handicap:

$$H(x, x_1^*, a) = \frac{1}{\ln[1 + f(x) - f(x_1^*)]} - a\|x - x_1^*\|^2, \tag{1.9}$$

where a is sufficiently large. Notice that the filled function in (1.9) is defined only for the region where $f(x) > f(x_1^*) - 1$. Xu et al [17] proposed a class of filled functions to deal with the handicap:

$$U(x, x_1^*, A, \beta) = -\eta(f(x) - f(x_1^*)) - A\omega(\|x - x_1^*\|^\beta), \tag{1.10}$$

where the functions $\eta(\cdot)$ and $\omega(\cdot)$ as well as the parameters A and β satisfy some conditions. However, the method for finding the function $\eta(\cdot)$ has not been specified in the paper. The concept of the filled function approach is promising. Its numerical performance, however, is far from a satisfaction. This consideration motivates the study reported in this paper.

This paper aims to develop a novel filled function with certain satisfactory properties in global optimization. This paper is organized as follows. Following this introduction, a new filled function is proposed in Section 2. The properties of the new filled function are investigated. In Section 3, numerical implementation is considered for the proposed new filled function and a solution algorithm is suggested. Application of the new filled-function algorithm to eight test problems is reported in Section 4 with satisfactory numerical results. Finally, some conclusions are drawn in Section 5.

2. A new filled function and its properties

We assume in this paper that the function $f(x)$ in (P) is Lipschitz continuous with constant L in $\mathbb{R}^n$. When a local minimizer, x_1^*, of $f(x)$ is found, the purpose of constructing a filled function is to move away from the current local minimizer, x_1^*, and to find a better minimizer, x_2^*, of $f(x)$ such that $f(x_2^*) < f(x_1^*)$, or to determine that the current local minimizer x_1^* is already a global minimizer of $f(x)$.

We propose in this paper a new two-parameter filled function for problem (P) at a local minimizer x_1^* as follows,

$$p(x, x_1^*, \rho, \mu) = f(x_1^*) - \min[f(x_1^*), f(x)] - \rho\|x - x_1^*\|^2 + \mu\{\max[0, f(x) - f(x_1^*)]\}^2. \tag{2.11}$$

When $f(x) \geqslant f(x_1^*)$,

$$p(x, x_1^*, \rho, \mu) = \mu[f(x) - f(x_1^*)]^2 - \rho\|x - x_1^*\|^2, \tag{2.12}$$

while when $f(x) \leqslant f(x_1^*)$,

$$p(x, x_1^*, \rho, \mu) = f(x_1^*) - f(x) - \rho\|x - x_1^*\|^2. \tag{2.13}$$

The following lemma and theorems show that $p(x, x_1^*, \rho, \mu)$ satisfies the conditions to be qualified as a filled function under some conditions on parameters ρ and μ. The following Lemma 2.1 is derived first to pave a way to prove that the local minimizer x_1^* is a strict maximizer of the filled function $p(x_1, x_1^*, \rho, \mu)$ under certain conditions.

Lemma 2.1. *Assume that x_1^* is a local minimizer of $f(x)$ and x_1 is a point such that $x_1 \neq x_1^*$ and $f(x_1) \geqslant f(x_1^*)$. If $\rho > 0$ and $0 \leqslant \mu < \frac{\rho}{L^2}$, then*

$$p(x_1, x_1^*, \rho, \mu) < 0 = p(x_1^*, x_1^*, \rho, \mu).$$

Proof. The following is evident from (2.12),

$$\begin{aligned}p(x_1,x_1^*,\rho,\mu) &= -\rho\|x_1-x_1^*\|^2+\mu[f(x_1)-f(x_1^*)]^2\\ &\leqslant -\rho\|x_1-x_1^*\|^2+\mu L^2\|x_1-x_1^*\|^2\\ &< 0=p(x_1^*,x_1^*,\rho,\mu).\end{aligned}$$

□

Theorem 2.1. *Assume that x_1^* is a local minimizer of $f(x)$. If $\rho>0$ and $0\leqslant\mu<\frac{\rho}{L^2}$, then x_1^* is a strict local maximizer of $p(x,x_1^*,\rho,\mu)$.*

Proof. Since x_1^* is a local minimizer of $f(x)$, there is a neighborhood $N(x_1^*,\sigma_1)$ of x_1^* with $\sigma_1>0$ such that $f(x_1)\geqslant f(x_1^*)$ for all $x_1\in N(x_1^*,\sigma_1)$. Then, by Lemma 2.1, for all $x_1\in N(x_1^*,\sigma_1)$, $x_1\neq x_1^*$,

$$p(x_1,x_1^*,\rho,\mu)<0=p(x_1^*,x_1^*,\rho,\mu).$$

Thus, x_1^* is a strict local maximizer of $p(x,x_1^*,\rho,\mu)$. □

Theorem 2.2. *Assume that x_1^* is a local minimizer of $f(x)$. Suppose that x_1 and x_2 are two points such that $\|x_1-x_1^*\|<\|x_2-x_1^*\|$ and $f(x_1^*)<f(x_1)<f(x_2)$. If $\rho>0$ and $0\leqslant\mu<\min\{\frac{\rho}{L^2},\frac{\rho}{LM}\}$, where*

$$M\geqslant\max_{0\leqslant\lambda\leqslant 1}\|\nabla f(x_1+\lambda(x_2-x_1))\|\frac{\|x_2-x_1\|}{\|x_2-x_1^*\|-\|x_1-x_1^*\|},$$

then

$$p(x_2,x_1^*,\rho,\mu)<p(x_1,x_1^*,\rho,\mu)<0=p(x_1^*,x_1^*,\rho,\mu).$$

Proof. Consider

$$\begin{aligned}&p(x_2,x_1^*,\rho,\mu)-p(x_1,x_1^*,\rho,\mu)\\ &=-\rho(\|x_2-x_1^*\|^2-\|x_1-x_1^*\|^2)\\ &+\mu\left\{[f(x_2)-f(x_1^*)]^2-[f(x_1)-f(x_1^*)]^2\right\}\\ &=(\|x_2-x_1^*\|^2-\|x_1-x_1^*\|^2)\cdot\\ &\left\{-\rho+\mu\frac{[f(x_2)-f(x_1^*)]^2-[f(x_1)-f(x_1^*)]^2}{\|x_2-x_1^*\|^2-\|x_1-x_1^*\|^2}\right\}\\ &=(\|x_2-x_1^*\|^2-\|x_1-x_1^*\|^2)\cdot\\ &\left\{-\rho+\mu\frac{[f(x_2)-f(x_1^*)+f(x_1)-f(x_1^*)][f(x_2)-f(x_1)]}{(\|x_2-x_1^*\|+\|x_1-x_1^*\|)(\|x_2-x_1^*\|-\|x_1-x_1^*\|)}\right\}\\ &\leqslant(\|x_2-x_1^*\|^2-\|x_1-x_1^*\|^2)\left\{-\rho+\mu L\frac{f(x_2)-f(x_1)}{\|x_2-x_1^*\|-\|x_1-x_1^*\|}\right\}\end{aligned}$$

$$
\begin{aligned}
&= (\|x_2 - x_1^*\|^2 - \|x_1 - x_1^*\|^2) \cdot \\
&\left\{ -\rho + \mu L \nabla^T f(x_1 + \lambda(x_2 - x_1)) \frac{x_2 - x_1}{\|x_2 - x_1\|} \frac{\|x_2 - x_1\|}{\|x_2 - x_1^*\| - \|x_1 - x_1^*\|} \right\} \\
&\leqslant (\|x_2 - x_1^*\|^2 - \|x_1 - x_1^*\|^2) \cdot \\
&\left\{ -\rho + \mu L \|\nabla f(x_1 + \lambda(x_2 - x_1))\| \frac{\|x_2 - x_1\|}{\|x_2 - x_1^*\| - \|x_1 - x_1^*\|} \right\} \\
&\leqslant (\|x_2 - x_1^*\|^2 - \|x_1 - x_1^*\|^2)(-\rho + \mu L M) \\
&< 0.
\end{aligned}
$$

Thus, the theorem follows directly from Lemma 2.1. □

We discuss now the existence of an upper bound of M in Theorem 2.2. Since $\nabla f(x)$ is continuous in X, hence there is a constant $M_1 > 0$ such that

$$
0 \leqslant \max_{0 \leqslant \lambda \leqslant 1} \|\nabla f(x_1 + \lambda(x_2 - x_1))\| \leqslant M_1, \quad \forall x_1, x_2 \in X.
$$

Furthermore, from $f(x_1^*) < f(x_1)$, $x_1 \neq x_1^*$. For given x_1 and x_2 such that $\|x_1 - x_1^*\| < \|x_2 - x_1^*\|$, there is a constant $\varepsilon > 0$ such that $\|x_2 - x_1^*\| - \|x_1 - x_1^*\| \geqslant \varepsilon$. Let $M_2 = \|x_2 - x_1\| / \varepsilon$, we have

$$
\frac{\|x_2 - x_1\|}{\|x_2 - x_1^*\| - \|x_1 - x_1^*\|} \leqslant M_2.
$$

Therefore, M can be selected to equal M_3, where $M_3 = M_1 M_2$.

Theorems 2.1–2.2 reveal that the proposed new filled function satisfies Property (P1).

To show that the proposed filled function satisfies Property (P2), the following theorem is derived first.

Theorem 2.3. *Assume that x_1^* is a local minimizer of $f(x)$. Suppose that x_1 and x_2 are two points such that $\|x_1 - x_1^*\| < \|x_2 - x_1^*\|$ and $f(x_1^*) \leqslant f(x_2) \leqslant f(x_1)$. If $\rho > 0$ and $0 \leqslant \mu < \frac{\rho}{L^2}$ then*

$$
p(x_2, x_1^*, \rho, \mu) < p(x_1, x_1^*, \rho, \mu) < 0 = p(x_1^*, x_1^*, \rho, \mu).
$$

Proof. Consider

$$
\begin{aligned}
&p(x_2, x_1^*, \rho, \mu) - p(x_1, x_1^*, \rho, \mu) \\
&= -\rho(\|x_2 - x_1^*\|^2 - \|x_1 - x_1^*\|^2) \\
&+ \mu \left\{ [f(x_2) - f(x_1^*)]^2 - [f(x_1) - f(x_1^*)]^2 \right\} \\
&< 0.
\end{aligned}
$$

The inequality in the theorem follows from Lemma 2.1. □

Theorem 2.4. *Assume that x_1^* is a local minimizer of $f(x)$. Suppose that x_1 is a point such that $f(x_1) > f(x_1^*)$. If $\rho > 0$ and $0 \leqslant \mu < \min\{\frac{\rho}{L^2}, \frac{\rho}{LM}\}$, then for any small $\varepsilon_1 > 0$, there exists d_1 such that $0 < \|d_1\| \leqslant \varepsilon_1$, $\|x_1 - d_1 - x_1^*\| < \|x_1 - x_1^*\| < \|x_1 + d_1 - x_1^*\|$, $f(x_1 \pm d_1) \geqslant f(x_1^*)$ and $p(x_1 + d_1, x_1^*, \rho, \mu) < p(x_1, x_1^*, \rho, \mu) < p(x_1 - d_1, x_1^*, \rho, \mu) < 0 = p(x_1^*, x_1^*, \rho, \mu)$.*

Proof. For a given $\varepsilon_1 > 0$, let

$$d_1 = \varepsilon_2 \frac{x_1 - x_1^*}{\|x_1 - x_1^*\|},$$

where $0 < \varepsilon_2 \leqslant \varepsilon_1$. Then $0 < \|d_1\| \leqslant \varepsilon_1$. Furthermore, if ε_1 is sufficiently small and the condition on M in Theorem 2.2 holds, then

$$\|x_1 + d_1 - x_1^*\| = (1+\varepsilon)\|x_1 - x_1^*\| > \|x_1 - x_1^*\|,$$

$$\|x_1 - d_1 - x_1^*\| = (1-\varepsilon)\|x_1 - x_1^*\| < \|x_1 - x_1^*\|,$$

$$f(x_1 \pm d_1) \geqslant f(x_1^*)$$

where $\varepsilon = \varepsilon_2 / \|x_1 - x_1^*\|$. Hence, if

$$\rho > 0 \quad \text{and} \quad 0 \leqslant \mu < \min\{\tfrac{\rho}{L^2}, \tfrac{\rho}{LM}\},$$

then the following holds from Theorem 2.2 and Theorem 2.3, $p(x_1 + d_1, x_1^*, \rho, \mu) < p(x_1, x_1^*, \rho, \mu) < p(x_1 - d_1, x_1^*, \rho, \mu) < 0 = p(x_1^*, x_1^*, \rho, \mu)$. □

The implication of Theorem 2.4 is clear: Any point x_1 with $f(x_1) > f(x_1^*)$ will never be a local minimizer of $p(x, x_1^*, \rho, \mu)$ when $\rho > 0$ and $0 \leqslant \mu < \min\{\frac{\rho}{L^2}, \frac{\rho}{LM}\}$. The following theorem further reinforces the above conclusion.

Theorem 2.5. *Assume that x_1^* is a local minimizer of $f(x)$. Suppose that x_1 is a point such that $f(x_1) > f(x_1^*)$. If $\rho > 0$ and if $\mu \geqslant 0$ is sufficiently small, then $\nabla p(x_1, x_1^*, \rho, \mu) \neq 0$, i.e., x_1 is not a stationary point of $p(x, x_1^*, \rho, \mu)$.*

Proof. When $f(x_1) > f(x_1^*)$,

$$\nabla p(x_1, x_1^*, \rho, \mu) = -2\rho(x_1 - x_1^*) + 2\mu[f(x_1) - f(x_1^*)]\nabla f(x_1).$$

Consider the following two cases:

1. If $\nabla f(x_1) = 0$, then $\nabla p(x_1, x_1^*, \rho, \mu) = -2\rho(x_1 - x_1^*) \neq 0$.
2. If $\nabla f(x_1) \neq 0$, let

$$d = \frac{x_1 - x_1^*}{\|x_1 - x_1^*\|} - \varepsilon \frac{\nabla f(x_1)}{\|\nabla f(x_1)\|},$$

where $\varepsilon > 0$ is sufficiently small. Then, the following holds,

$$\begin{aligned}
&\nabla^T p(x_1, x_1^*, \rho, \mu) d \\
&= -2\rho\|x_1 - x_1^*\| + 2\rho\varepsilon(x_1 - x_1^*)^T \frac{\nabla f(x_1)}{\|\nabla f(x_1)\|} \\
&+2\mu[f(x_1) - f(x_1^*)]\nabla^T f(x_1) \frac{x_1 - x_1^*}{\|x_1 - x_1^*\|} \\
&-2\mu\varepsilon[f(x_1) - f(x_1^*)]\|\nabla f(x_1)\|.
\end{aligned}$$

(a) If $(x_1 - x_1^*)^T \nabla f(x_1) \leqslant 0$, then $\nabla^T p(x_1, x_1^*, \rho, \mu) d < 0$ and therefore $\nabla p(x_1, x_1^*, \rho, \mu) \neq 0$.

(b) If $(x_1 - x_1^*)^T \nabla f(x_1) > 0$, then choose $\mu \geqslant 0$ to be sufficiently small. Since $\varepsilon > 0$ is chosen to be sufficiently small, thus $\nabla^T p(x_1, x_1^*, \rho, \mu) d < 0$. Therefore, $\nabla p(x_1, x_1^*, \rho, \mu) \neq 0$.

We can conclude that any point x_1 with $f(x_1) > f(x_1^*)$ will never be a stationary point of $p(x, x_1^*, \rho, \mu)$ when $\rho > 0$ and μ is very small. □

Theorem 2.6. *Assume that x_1^* is a local minimizer of $f(x)$. If $\rho > 0$ and if $\mu \geqslant 0$ is sufficiently small, then any local minimizer or saddle point of $p(x, x_1^*, \rho, \mu)$ must belong to the set $S = \{x \in \mathbb{R}^n : f(x) \leqslant f(x_1^*)\}$.*

Proof. If the theorem is not true, then there is a local minimizer or saddle point of $p(x, x_1^*, \rho, \mu)$, $\bar{x}_1^*$, such that $\bar{x}_1^* \notin S$ and $f(\bar{x}_1^*) > f(x_1^*)$. Since x_1^* is a strict local maximizer of $p(x, x_1^*, \rho, \mu)$ and $\bar{x}_1^*$ is a local minimizer or saddle point of $p(x, x_1^*, \rho, \mu)$, thus $x_1^* \neq \bar{x}_1^*$. If $\bar{x}_1^*$ is a local minimizer of $p(x, x_1^*, \rho, \mu)$, it contradicts Theorem 2.4 when $\rho > 0$ and $0 \leqslant \mu < \min\{\frac{\rho}{L^2}, \frac{\rho}{LM}\}$. Similarly, if $\bar{x}_1^*$ is a saddle point of $p(x, x_1^*, \rho, \mu)$, it contradicts Theorem 2.5 when $\rho > 0$ and $\mu \geqslant 0$ is sufficiently small. Consequently, the theorem is true. □

In summary, the filled function proposed in (2.11) does satisfy property (P2) that it has no minimizers or saddle points in any higher basin of $f(x)$.

Theorem 2.7. *Assume that x_1^* is a local minimizer of $f(x)$. If x_2^* is another local minimizer of $f(x)$ satisfying $f(x_2^*) < f(x_1^*)$, then there is a neighborhood $N(x_2^*, \sigma_2)$ of x_2^* with $\sigma_2 > 0$ such that $p(x, x_1^*, \rho, \mu)$ has a minimizer, x', on the line segment connecting x_1^* and x_2 for every $x_2 \in N(x_2^*, \sigma_2)$ when $0 \leqslant \mu < \frac{\rho}{L^2}$ and $0 < \rho < \frac{\varepsilon_1}{D_1}$ where $0 < \varepsilon_1 < f(x_1^*) - f(x_2)$ and $D_1 = \max_{x \in N(x_2^*, \sigma_2)} \|x - x_1^*\|^2$. Moreover, if there does not exist any basin lower than B_1^* between B_1^* and B_2^*, where B_1^* and B_2^* are the basins of $f(x)$ at x_1^* and x_2^*, respectively, then $x' \in B_2^*$ and $f(x') \leqslant f(x_1^*)$.*

Proof. From Theorem 2.1, there is a neighborhood $N(x_1^*, \sigma_1)$ of x_1^* with $\sigma_1 > 0$ such that for all $x_1 \in N(x_1^*, \sigma_1)$ and $x_1 \neq x_1^*$, we have the following when $\rho > 0$ and $0 \leqslant \mu < \frac{\rho}{L^2}$,

$$p(x_1, x_1^*, \rho, \mu) < 0 = p(x_1^*, x_1^*, \rho, \mu).$$

Furthermore, there is a neighborhood $N(x_2^*, \sigma_2)$ of x_2^* with $\sigma_2 > 0$ such that $f(x_1^*) - f(x_2) > \varepsilon_1 > 0$ for all $x_2 \in N(x_2^*, \sigma_2)$. Thus,

$$p(x_2, x_1^*, \rho, \mu) = f(x_1^*) - f(x_2) - \rho\|x_2 - x_1^*\|^2 > \varepsilon_1 - \rho D_1.$$

Therefore, if $\rho < \frac{\varepsilon_1}{D_1}$ then $p(x_2, x_1^*, \rho, \mu) > 0$. The filled function $p(x, x_1^*, \rho, \mu)$ is decreasing along the line connecting x_1^* and x_2 when starting from x_1^*. The continuity of $p(x, x_1^*, \rho, \mu)$ implies that $p(x, x_1^*, \rho, \mu)$ has a minimizer on the line segment connecting x_1^* and x_2 for all $x_2 \in N(x_2^*, \sigma_2)$.

Let x_B be the boundary point of B_2^* on the line segment. If there does not exist any basin lower than B_1^* between B_1^* and B_2^*, then $f(x_B) > f(x_1^*)$. Thus, by continuity of $f(x)$, there are three points $x_0^-, x_0, x_0^+ \in B_2^*$ on the line segment such that $f(x_0) = f(x_1^*)$ and $f(x_B) > f(x_0^-) \geqslant f(x_0) \geqslant f(x_0^+) > f(x_2)$ where $x_0^- = x_0 - \sigma(x_0 - x_1^*)$ and $x_0^+ = x_0 + \sigma(x_0 - x_1^*)$ provided $\sigma > 0$ is sufficiently small. Since $p(x_0, x_1^*, \rho, \mu) = -\rho\|x_0 - x_1^*\|^2 < 0$, hence $x_0 \notin N(x_2^*, \sigma_2)$ from the previous discussion. Now, we consider the following two cases:

1. If $p(x_0, x_1^*, \rho, \mu) > p(x_0^+, x_1^*, \rho, \mu)$, then $x_0 - x_1^*$ is a decent direction of $p(x, x_1^*, \rho, \mu)$ at x_0. Therefore, $x' \in B_2^*$ and $f(x') \leqslant f(x_1^*)$.
2. If $p(x_0, x_1^*, \rho, \mu) \leqslant p(x_0^+, x_1^*, \rho, \mu)$, since $\|x_0^- - x_1^*\| < \|x_0 - x_1^*\|$ and $f(x_0^-) \geqslant f(x_0)$, thus, by Theorem 2.3, $0 > p(x_0^-, x_1^*, \rho, \mu) > p(x_0, x_1^*, \rho, \mu)$ when $\rho > 0$ and $0 \leqslant \mu < \frac{\rho}{L^2}$. Hence, $x_0 - x_1^*$ is a decent direction of $p(x, x_1^*, \rho, \mu)$ at x_0^-. Therefore, $x' \equiv x_0 \in B_2^*$ and $f(x') = f(x_1^*)$.

□

Theorems 2.6–2.7 clearly state that the proposed filled function satisfies Property (P3). Moreover, the filled function proposed in this paper has a prominent feature: Unlike the filled function suggested in Ge [5] and Liu [11], our filled function guarantees its minimum to be always achieved at a point where its function value is not higher than the function value of the current minimum.

Theorem 2.8. *Assume that x_1^* is a global minimizer of $f(x)$. If $\rho > 0$ and $0 \leqslant \mu < \frac{\rho}{L^2}$ then $p(x, x_1^*, \rho, \mu) < 0$ for all $x \in X$.*

Proof. Since x_1^* is a global minimizer of $f(x)$, $f(x) \geqslant f(x_1^*)$ for all $x \in X$. Thus, by Lemma 2.1, $p(x, x_1^*, \rho, \mu) < 0$ for all $x \in X$ when $\rho > 0$ and $0 \leqslant \mu < \frac{\rho}{L^2}$. □

3. Numerical implementation and solution algorithm

The theoretical properties of the proposed filled function $p(x, x_1^*, \rho, \mu)$ were discussed in the last section. The development in this section is to further study the properties of the proposed filled function in numerical implementation and to suggest an efficient algorithm. Three search directions are investigated first such that one can start at an initial point $x_1^{(0)} \in X \setminus N(x_1^*, \sigma_1)$ for some $\sigma_1 > 0$ to escape from the current local minimum x_1^* and to minimize the proposed filled function along the search directions in order to reach a point $x_1^{(k)}$ with $f(x_1^{(k)}) < f(x_1^*)$. Then, one can use $x_1^{(k)}$ as an initial point in a local search to find a better local minimizer x_2^* with $f(x_2^*) < f(x_1^*)$. The algorithm is repeated in a two-phase iterative fashion until a global solution is identified (no better local solution can be found).

3.1. SEARCH DIRECTIONS IN NUMERICAL IMPLEMENTATION

How to decide a search direction in finding another better local minimum is a key to the success in a filled function approach. Let x_1^* be the current local minimum, $x_1^{(i)}$ the current iterative point, and $d_1^{(i)}$ the search direction. The discussion in this subsection will lead to a conclusion that a good search direction should satisfy $(x_1^{(i)} - x_1^*)^T d_1^{(i)} > 0$.

Theorem 3.1. *For two given constants λ_L and λ_U with $0 < \lambda_L < \lambda_U$, let $x_1^{(i)} \in X$ and $x_1^{(i+1)} = x_1^{(i)} + d_1^{(i)} \in X$ where $d_1^{(i)}$ is a search direction at $x_1^{(i)}$ such that $\lambda_L \leqslant \|d_1^{(i)}\| \leqslant \lambda_U$. Let $\theta_1^{(i)}$ be the angle between $x_1^{(i)} - x_1^*$ and $d_1^{(i)}$. Then the following are equivalent.*

1. $\|x_1^{(i+1)} - x_1^*\| > \|x_1^{(i)} - x_1^*\|$.
2. $2(x_1^{(i)} - x_1^*)^T d_1^{(i)} + \|d_1^{(i)}\|^2 > 0$.
3. $\cos\theta_1^{(i)} > -\dfrac{\|d_1^{(i)}\|}{2\|x_1^{(i)} - x_1^*\|}$.
4. $(x_1^{(i)} - x_1^*)^T d_1^{(i)} + (x_1^{(i+1)} - x_1^*)^T d_1^{(i)} > 0$.

In particular, if $(x_1^{(k)} - x_1^*)^T d_1^{(k)} \geqslant 0, \forall k = 0, 1, \ldots, i-1$, then $\|x_1^{(i)} - x_1^*\|^2 \geqslant i\lambda_L^2 + \|x_1^{(0)} - x_1^*\|^2$.

Proof. Observe the following equalities:

$$\begin{aligned}\|x_1^{(i+1)} - x_1^*\|^2 - \|x_1^{(i)} - x_1^*\|^2 &= \|x_1^{(i)} - x_1^* + d_1^{(i)}\|^2 - \|x_1^{(i)} - x_1^*\|^2 \\ &= 2(x_1^{(i)} - x_1^*)^T d_1^{(i)} + \|d_1^{(i)}\|^2;\end{aligned}$$

$$2(x_1^{(i)} - x_1^*)^T d_1^{(i)} + \|d_1^{(i)}\|^2 = (2\|x_1^{(i)} - x_1^*\| \cos\theta_1^{(i)} + \|d_1^{(i)}\|)\|d_1^{(i)}\|;$$

$$\begin{aligned}2(x_1^{(i)} - x_1^*)^T d_1^{(i)} + \|d_1^{(i)}\|^2 &= [(x_1^{(i)} - x_1^*) + (x_1^{(i)} - x_1^* + d_1^{(i)})]^T d_1^{(i)} \\ &= (x_1^{(i)} - x_1^*)^T d_1^{(i)} + (x_1^{(i+1)} - x_1^*)^T d_1^{(i)}.\end{aligned}$$

Hence the following are equivalent:

$$0 < \|x_1^{(i+1)} - x_1^*\|^2 - \|x_1^{(i)} - x_1^*\|^2;$$
$$0 < 2(x_1^{(i)} - x_1^*)^T d_1^{(i)} + \|d_1^{(i)}\|^2;$$
$$0 < (2\|x_1^{(i)} - x_1^*\| \cos\theta_1^{(i)} + \|d_1^{(i)}\|)\|d_1^{(i)}\|;$$
$$0 < (x_1^{(i)} - x_1^*)^T d_1^{(i)} + (x_1^{(i+1)} - x_1^*)^T d_1^{(i)}.$$

Consequently, Items 1–4 are equivalent.

In particular, if $(x_1^{(k)} - x_1^*)^T d_1^{(k)} \geqslant 0, \forall k = 0, 1, \ldots, i-1$, then $2(x_1^{(k)} - x_1^*)^T d_1^{(k)} + \|d_1^{(k)}\|^2 > 0$ and hence $\|x_1^{(k+1)} - x_1^*\| > \|x_1^{(k)} - x_1^*\|$. Therefore,

$$\begin{aligned}
&\|x_1^{(i)} - x_1^*\|^2 - \|x_1^{(0)} - x_1^*\|^2 \\
&= (\|x_1^{(i)} - x_1^*\|^2 - \|x_1^{(i-1)} - x_1^*\|^2) + (\|x_1^{(i-1)} - x_1^*\|^2 - \|x_1^{(i-2)} - x_1^*\|^2) \\
&+ \cdots + (\|x_1^{(1)} - x_1^*\|^2 - \|x_1^{(0)} - x_1^*\|^2) \\
&= \left[2(x_1^{(i-1)} - x_1^*)^T d_1^{(i-1)} + \|d_1^{(i-1)}\|^2\right] \\
&+ \left[2(x_1^{(i-2)} - x_1^*)^T d_1^{(i-2)} + \|d_1^{(i-2)}\|^2\right] \\
&+ \cdots + \left[2(x_1^{(0)} - x_1^*)^T d_1^{(0)} + \|d_1^{(0)}\|^2\right] \\
&\geqslant i\lambda_L^2.
\end{aligned}$$

□

From Theorem 3.1, we can conclude that if a search direction $d_1^{(i)}$ is chosen to satisfy $(x_1^{(i)} - x_1^*)^T d_1^{(i)} \geqslant 0$, the search will reach the boundary of X when the number of iterations is sufficiently large provided no better point in a lower basin is found before that happens.

Theorem 3.2. *Let $d \neq 0$ be a search direction at $x \in X$ where $f(x) > f(x_1^*)$. Suppose that $\rho > 0$ and $\mu \geqslant 0$. Then $d^T \nabla p(x, x_1^*, \rho, \mu) < 0$ if and only if one of the following conditions holds*:

1. $d^T(x - x_1^*) > 0$ *and* $\mu = 0$.
2. $d^T(x - x_1^*) > 0$, $d^T \nabla f(x) > 0$ *and* $\mu < \rho d^T(x - x_1^*)/\{[f(x) - f(x_1^*)] d^T \nabla f(x)\}$.
3. $d^T(x - x_1^*) > 0$ *and* $d^T \nabla f(x) \leqslant 0$.
4. $d^T(x - x_1^*) = 0$, $d^T \nabla f(x) < 0$ *and* $\mu > 0$.
5. $d^T(x - x_1^*) < 0$, $d^T \nabla f(x) < 0$ *and* $\mu > \rho d^T(x - x_1^*)/\{[f(x) - f(x_1^*)] d^T \nabla f(x)\}$.

Proof. We have

$$\begin{aligned}
0 &> d^T \nabla p(x, x_1^*, \rho, \mu) \\
&= -2\rho d^T(x - x_1^*) + 2\mu[f(x) - f(x_1^*)] d^T \nabla f(x).
\end{aligned}$$

Since $\rho > 0$ and $\mu \geqslant 0$, the theorem follows from the fact that the 5 listed mutually exclusive situations cover all the circumstances for $d^T \nabla p(x, x_1^*, \rho, \mu) < 0$. □

Theorem 3.2 lists all possible circumstances for descent directions of $p(x, x_1^*, \rho, \mu)$. It is easy to see that these search directions d satisfying $d^T(x - x_1^*) \leqslant 0$ are more restrictive than those with $d^T(x - x_1^*) > 0$. More specifically, no descent direction of $p(x, x_1^*, \rho, \mu)$ can satisfy both $d^T(x - x_1^*) \leqslant 0$ and $d^T \nabla f(x) \geqslant 0$ at the same time. We can conclude that these search directions d satisfying $d^T(x - x_1^*) > 0$ are better candidates to serve as descent directions for $p(x, x_1^*, \rho, \mu)$, since they are always applicable no matter whether $d^T \nabla f(x)$ is positive or negative.

In the following, three specific search directions and their properties are investigated.

Search direction D_1 and its properties

Let $D_1 = x - x_1^*$ be a search direction at $x \in X$ where $f(x) > f(x_1^*)$. Then D_1 has the following properties:

1. $D_1^T(x - x_1^*) > 0$.
2. From Conditions 1–3 of Theorem 3.2, if $\rho > 0$, $\mu \geqslant 0$, $D_1^T \nabla f(x) > 0$ and $D_1^T \nabla p(x, x_1^*, \rho, \mu) \geqslant 0$, then μ has been selected to be too large

$$(\mu \geqslant \frac{\rho D_1^T(x - x_1^*)}{[f(x) - f(x_1^*)] D_1^T \nabla f(x)}).$$

Search direction D_2 and its properties

Let $D_2 = -\nabla p(x, x_1^*, \rho, \mu)$ be a search direction at $x \in X$ where $\rho > 0$, $\mu \geqslant 0$ and $f(x) > f(x_1^*)$. Then D_2 has the following properties:

1. If $D_2 = 0$, then we can conclude from Theorem 2.2 and Theorem 2.3 that μ has not been selected sufficiently small.
2. If $D_2 \neq 0$, then $D_2^T \nabla p(x, x_1^*, \rho, \mu) < 0$. Therefore, one of the conditions listed in Theorem 3.2 holds.
3. If $\mu = 0$, then $D_2 = 2\rho D_1$.

Search direction D_3 and its properties

Let

$$D_3 = -\frac{\nabla f(x)}{\|\nabla f(x)\|} - \frac{\nabla p(x, x_1^*, \rho, \mu)}{\|\nabla p(x, x_1^*, \rho, \mu)\|}$$

be a search direction at $x \in X$ where $\rho > 0$, $\mu \geqslant 0$, $f(x) > f(x_1^*)$, $\|\nabla f(x)\| \neq 0$ and $\|\nabla p(x, x_1^*, \rho, \mu)\| \neq 0$.

Theorem 3.3. *If $D_3 \neq 0$, then D_3 has the following properties*:

1. $D_3^T \nabla f(x) < 0$ *and* $D_3^T \nabla p(x, x_1^*, \rho, \mu) < 0$.
2. *Exact one of the following conditions holds*:

   (*a*) $D_3^T(x - x_1^*) > 0$.
   (*b*) $D_3^T(x - x_1^*) = 0$ *and* $\mu > 0$.
   (*c*) $D_3^T(x - x_1^*) < 0$ *and* $\mu > \frac{\rho D_3^T(x-x_1^*)}{[f(x)-f(x_1^*)]D_3^T \nabla f(x)}$.

Proof.

1. Let θ be the angle between $\nabla f(x)$ and $\nabla p(x, x_1^*, \rho, \mu)$. Condition $D_3 = 0$ implies

$$0 = \frac{\nabla f(x)}{\|\nabla f(x)\|} + \frac{\nabla p(x, x_1^*, \rho, \mu)}{\|\nabla p(x, x_1^*, \rho, \mu)\|}$$

which is equivalent to

$$-1 = \frac{\nabla^T f(x) \nabla p(x, x_1^*, \rho, \mu)}{\|\nabla f(x)\| \|\nabla p(x, x_1^*, \rho, \mu)\|} = \cos\theta.$$

Thus, $D_3 \neq 0$ is equivalent to $\cos\theta \neq -1$.
Therefore,

$$\begin{aligned}\nabla^T f(x) D_3 &= -\|\nabla f(x)\| - \tfrac{\nabla^T f(x) \nabla p(x,x_1^*,\rho,\mu)}{\|\nabla p(x,x_1^*,\rho,\mu)\|}\\ &= -\|\nabla f(x)\|(1+\cos\theta) < 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\nabla^T p(x, x_1^*, \rho, \mu) D_3 &= -\tfrac{\nabla^T p(x,x_1^*,\rho,\mu)\nabla f(x)}{\|\nabla f(x)\|} - \|\nabla p(x, x_1^*, \rho, \mu)\|\\ &= -\|\nabla p(x, x_1^*, \rho, \mu)\|(\cos\theta + 1) < 0.\end{aligned}$$

2. If $D_3 \neq 0$, then $D_3^T \nabla p(x, x_1^*, \rho, \mu) < 0$ is equivalent to

$$-\rho D_3^T(x - x_1^*) + \mu[f(x) - f(x_1^*)] D_3^T \nabla f(x) < 0$$

where $D_3^T \nabla f(x) < 0$. By listing all the combinations of the terms, the conditions are clear.

□

3.2. ALGORITHM

Based on the results in the previous subsection, a solution algorithm is proposed as follows.

1. Starting from an initial point $x_1 \in X$, minimize $f(x)$ and obtain the first local minimizer x_1^* of $f(x)$.
2. Initialization:
   (a) Choose a tolerance $\varepsilon > 0$, e.g., set $\varepsilon := 10^{-4}$.
   (b) Choose a lower bound of ρ such that $\rho_L > 0$, e.g., set $\rho_L := 10^{-3}$.
   (c) The lower bound of μ is 0. Choose a second lower bound of μ such that $\mu_L > 0$, e.g., set $\mu_L := 10^{-8}$.
   (d) Choose a fraction $\hat{\rho} > 0$, e.g., set $\hat{\rho} := 0.1$.
   (e) Choose a fraction $\hat{\mu} > 0$, e.g., set $\hat{\mu} := 0.1$.
   (f) Choose $\rho > 0$, e.g., set $\rho := 1$.
   (g) Set $k := 1$.
3. Choose a set of initial points $\{x_{k+1}^{(0)i} : i = 1, 2, \ldots, m\}$ such that $x_{k+1}^{(0)i} \in X \setminus N(x_k^*, \sigma_k)$ for some $\sigma_k > 0$.
4. Choose $\mu \geqslant 0$, e.g., set $\mu := \rho$.
5. Set $i := 1$.
6. (a) If $i \leqslant m$, then set $x := x_{k+1}^{(0)i}$ and go to 7.
   (b) Otherwise, go to 14.
7. (a) If $f(x) < f(x_k^*)$, then use x as an initial point for a local minimization method to find x_{k+1}^* such that $f(x_{k+1}^*) < f(x_k^*)$. Then, set $k := k+1$ and go to 3.
   (b) Otherwise, go to 8.
8. (a) If $\|\nabla p(x, x_k^*, \rho, \mu)\| \geqslant n\varepsilon$, go to 9.
   (b) Otherwise, go to 13.
9. If all the following conditions hold, select

$$D_3 = -\frac{\nabla f(x)}{\|\nabla f(x)\|} - \frac{\nabla p(x, x_k^*, \rho, \mu)}{\|\nabla p(x, x_k^*, \rho, \mu)\|}$$

   as the search direction and go to 10; otherwise, go to 11.
   (a) $\|\nabla f(x)\| \geqslant n\varepsilon$;
   (b) $\|D_3\| \geqslant n\varepsilon$;
   (c) $D_3^T(x - x_k^*) \geqslant 0$;
   (d) $\nabla^T f(x) \nabla p(x, x_k^*, \rho, \mu) > 0$.
10. Find a new x in the direction D_3 such that both $p(x, x_k^*, \rho, \mu)$ and $f(x)$ can reduce to certain extents.
    (a) If x attains the boundary of X during minimization, then set $i := i+1$ and go to 6.
    (b) Otherwise, go to 7.
11. (a) If $D_2^T(x - x_k^*) \geqslant 0$ where $D_2 = -\nabla p(x, x_k^*, \rho, \mu)$, then select D_2 as the search direction and go to 12.
    (b) Otherwise, go to 13.
12. Find a new x in the direction D_2 such that $p(x, x_k^*, \rho, \mu)$ can reduce to certain extent.

   (a) If x attains the boundary of X during minimization, then set $i := i+1$ and go to 6.
   (b) Otherwise, go to 7.

13. Reduce μ by setting $\mu := \hat{\mu}\mu$.
   (a) If $\mu \geqslant \mu_L$, then go to 5.
   (b) Otherwise, set $\mu := 0$ and go to 5.
14. Reduce ρ by setting $\rho := \hat{\rho}\rho$.
   (a) If $\rho \geqslant \rho_L$, then go to 4.
   (b) Otherwise, the algorithm is incapable of finding a better minimizer starting from the initial points, $\{x_{k+1}^{(0)i}\}$. The algorithm stops and x_k^* is taken as a global minimizer.

The motivation and mechanism behind the algorithm are explained below.

A set of m initial points is chosen in Step 3 to minimize the filled function. If no additional information about the objective function is provided, we set the initial points symmetric about the current local minimizer. For example, when $n=2$, the initial points are: $x_k^* + \delta \times (\cos(2(i-1)\pi/m), \sin(2(i-1)\pi/m))$, where $\delta > 0$ is a pre-selected step-size.

Step 7 represents a transition from minimizing the filled function $p(x, x_k^*, \rho, \mu)$ to a local search for the original objective function $f(x)$ when a point x with $f(x) < f(x_k^*)$ is found. It can be concluded from Theorem 2.7 that this point x must be in a lower basin of $f(x)$. We can thus use x as the initial point to minimize $f(x)$ in this lower basin and obtain a better local minimizer.

Step 8 guarantees that $\nabla p(x, x_k^*, \rho, \mu) \neq 0$ when $f(x) > f(x_k^*)$. If $\nabla p(x, x_k^*, \rho, \mu) = 0$, it can be concluded from Theorem 2.5 that μ is not chosen small enough. The algorithm then goes to Step 13 and the value of μ is reduced.

Step 9 checks if D_3 is a more desirable search direction than D_2. First, Items 9(a)–(b) ensure that D_3 is definite (based on the definition of D_3). Then, Item 9(c) guarantees that the search will reach the boundary of X when the number of iterations is sufficiently large provided no better point in a lower basin is found before that happens (Theorem 3.1). Finally, Item 9(d) ensures that the angle between $\nabla f(x)$ and $\nabla p(x, x_k^*, \rho, \mu)$, θ, is acute. From Theorem 3.3, a property of D_3 is $D_3^T \nabla f(x) < 0$. This implies that D_3 should not be used as a direction to escape from a basin. If the search is entering a basin, then θ must be an acute angle and D_3 would be a good search direction. On the contrary, if the search is leaving a basin, then θ must be an obtuse angle and D_3 is not a search direction as good as D_2.

If D_3 is selected as the search direction in Step 9, then there is a new point in the direction D_3 such that both $p(x, x_k^*, \rho, \mu)$ and $f(x)$ can be reduced to certain extent (Property 1 of D_3). Step 10 finds such a new x so that it can be used to minimize the filled function in a recursive fashion.

However, if D_3 is not selected as the search direction in Step 9, Step 11 ensures that the search along D_2 will reach the boundary of X when the number of iterations is sufficiently large provided no better point in a lower basin is found before that happens (Theorem 3.1).

Step 12 finds a new x in the direction D_2 such that $p(x, x_k^*, \rho, \mu)$ can reduce to certain extent (Property 2 of D_2). Then, one can use the new x to minimize the filled function again.

Recall from Theorem 2.7 that the value of ρ should be selected small enough. Otherwise, there could be no minimizer of $p(x, x_k^*, \rho, \mu)$ in a better basin. Thus, the value of ρ is reduced successively in Step 14 of the solution process if no better solution is found when minimizing the filled function. If the value of ρ reaches its lower bound ρ_L and no better solution is found, the current local minimizer is taken as a global minimizer.

Similar argument applies to μ. The value of μ should be selected small enough (less than $\min\{\frac{\rho}{L^2}, \frac{\rho}{LM}\}$). Thus, the value of μ is reduced successively in Step 13 of the solution process when certain step fails. When μ needs to be reduced further at μ_L, μ is assigned to zero.

4. Numerical experiment

In this section, the proposed algorithm is examined on eight test problems taken from the literature. A set of Matlab programs is written on UNIX platform to implement the tests. The Matlab function 'fmincon' is used in the algorithm to find local minimizers of the objective function.

The computational results are summarized in tables for each example problem. The symbols used in the tables are given as follows:

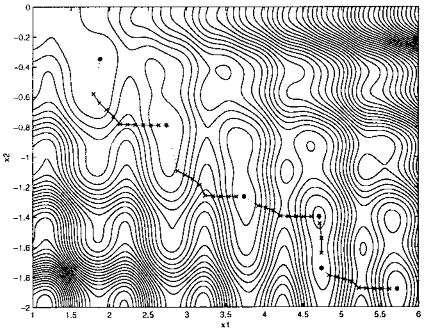
- k: The iteration number in finding the k-th local minimizer
- $x_k^{(0)}$: The starting point in the k-th iteration in finding the k-th local minimizer. Specifically, the starting point in the $(k+1)$-th iteration, $x_{k+1}^{(0)}$, is equal to the local minimizer x_k^* achieved in the k-th iteration plus a small perturbation specified in Step 3 of the algorithm. We only display the successful initial point in the tables which leads to the minimizer in the $(k+1)$-th iteration.
- ρ, μ: The parameters used for finding the k-th local minimizer
- x_k^*: The k-th local minimizer
- $f(x_k^*)$: The function value of the k-th local minimizer

Problem 1 (Two-dimensional function in [19]).

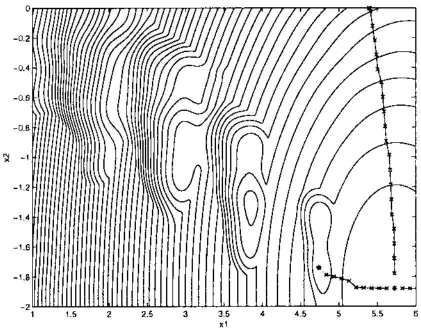
$$\min \ f(x) = [1 - 2x_2 + c\sin(4\pi x_2) - x_1]^2 + [x_2 - 0.5\sin(2\pi x_1)]^2,$$
$$\text{s.t.} \ 0 \leqslant x_1 \leqslant 10, \quad -10 \leqslant x_2 \leqslant 0,$$

where $c = 0.2, 0.5, 0.05$.

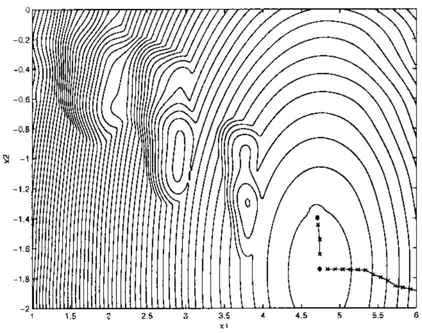
The proposed filled function approach succeeds in identifying the global minimum solutions: $f(x^*) = 0$ for all c, with $m = 4$ and $\delta = 0.1$. The computational results are summarized in Tables I–III, for $c = 0.2, 0.5, 0.05$, respectively.



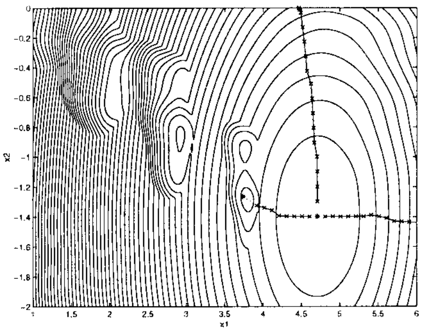
(a) Contours of $f(x)$ and the successful search-path from x_1^* to x_6^*



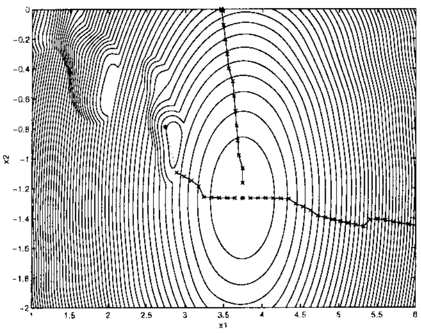
(b) Contours of $p(x, x_1^*, 1, 1\text{e-}3)$ and all the search-paths from x_1^*



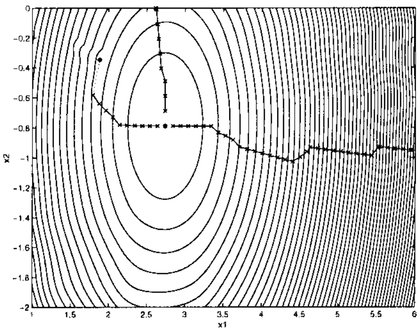
(c) Contours of $p(x, x_2^*, 1, 1\text{e-}3)$ and all the search-paths from x_2^*



(d) Contours of $p(x, x_3^*, 1, 1\text{e-}3)$ and all the search-paths from x_3^*



(e) Contours of $p(x, x_4^*, 1, 1\text{e-}3)$ and all the search-paths from x_4^*



(f) Contours of $p(x, x_5^*, 1, 1\text{e-}3)$ and all the search-paths from x_5^*

Figure 1. Contours and search-paths of Problem 1 with $c=0.2$

A detailed search process of Problem 1 with $c=0.2$ is displayed in Figure 1. The symbols used in the figure are given as follows:

$*$:	Local minimizer
$\times$——$\times$:	Search-path of the filled function
$\times\cdots*$:	Local search using "fmincon"

Problem 2 (Three-hump back camel function in [5]).

$$\begin{aligned}&\min\ f(x)=2x_1^2-1.05x_1^4+\frac{1}{6}x_1^6-x_1x_2+x_2^2,\\&\text{s.t.}\ -3\leqslant x_1\leqslant 3,\quad -3\leqslant x_2\leqslant 3.\end{aligned}$$

Two initial points $x=(-2,-1)$ and $(2,1)$ are used. The proposed filled function approach succeeds in identifying the global minimum solution: $x^*=(0,0)$ and $f(x^*)=0$, with $m=4$ and $\delta=0.1$. The computational results are summarized in Tables IV and V, respectively.

Problem 3 (Six-hump back camel function in [5]).

$$\begin{aligned}&\min\ f(x)=4x_1^2-2.1x_1^4+\frac{1}{3}x_1^6-x_1x_2-4x_2^2+4x_2^4,\\&\text{s.t.}\ -3\leqslant x_1\leqslant 3,\quad -3\leqslant x_2\leqslant 3.\end{aligned}$$

Three initial points $x=(-2,1)$, $(2,-1)$, and $(-2,-1)$ are used. The proposed filled function approach succeeds in identifying the global minimum solutions:

$$\begin{aligned}&x^*=(0.0898420131003, 0.712656403021)\\&\text{or } (-0.0898420131003, -0.712656403021)\end{aligned}$$

where $f(x^*)=-1.03162845349$, with $m=4$ and $\delta=0.1$. The computational results are summarized in Tables VI–VIII, respectively.

Problem 4 (Treccani function in [5]).

$$\begin{aligned}&\min\ f(x)=x_1^4+4x_1^3+4x_1^2+x_2^2,\\&\text{s.t.}\ -3\leqslant x_1\leqslant 3,\quad -3\leqslant x_2\leqslant 3.\end{aligned}$$

The proposed filled function approach succeeds in identifying a global minimum solution: $x^*=(0,0)$ where $f(x^*)=0$, with $m=4$ and $\delta=0.1$. The computational results are summarized in Table IX.

Problem 5 (Goldstein and Price function in [5]).

$$\begin{aligned}&\min\ f(x)=g(x)h(x),\\&\text{s.t.}\ -3\leqslant \mathrm{x}_1\leqslant 3,\quad -3\leqslant \mathrm{x}_2\leqslant 3,\end{aligned}$$

where

$$\begin{aligned}&g(x)=1+(x_1+x_2+1)^2(19-14x_1+3x_1^2-14x_2+6x_1x_2+3x_2^2),\\&h(x)=30+(2x_1-3x_2)^2(18-32x_1+12x_1^2+48x_2-36x_1x_2+27x_2^2).\end{aligned}$$

The proposed filled function approach succeeds in identifying the global minimum solution: $x^*=(0,-1)$ and $f(x^*)=3$, with $m=4$ and $\delta=0.1$. The computational results are summarized in Table X.

Problem 6 (Two-dimensional Shubert function in [5]).

$$\min f(x)=\left\{\sum_{i=1}^{5} i\cos[(i+1)x_1+i]\right\}\left\{\sum_{i=1}^{5} i\cos[(i+1)x_2+i]\right\},$$
$$\text{s.t. } 0\leqslant x_1\leqslant 10,\quad 0\leqslant x_2\leqslant 10.$$

The proposed filled function approach succeeds in identifying the global minimum solutions: $x^*=(5.48286420671, 4.85805687886)$ or $(4.85805687886, 5.48286420671)$ where $f(x^*)=-186.730908831$, with $m=4$ and $\delta=0.1$. The computational results are summarized in Table XI.

Problem 7 (Shekel's function in [12]).

$$\min \quad f(x)=-\sum_{i=1}^{5}\left[\sum_{j=1}^{4}(x_j-a_{i,j})^2+c_i\right]^{-1},$$
$$\text{s.t. } 0\leqslant x_j\leqslant 10,\quad j=1,2,3,4,$$

where the coefficients $a_{i,j}$, c_i, $i=1,2,3,4,5$, $j=1,2,3,4$ are given in the following:

i	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$a_{i,4}$	c_i
1	*4.0*	*4.0*	*4.0*	*4.0*	*0.1*
2	*1.0*	*1.0*	*1.0*	*1.0*	*0.2*
3	*8.0*	*8.0*	*8.0*	*8.0*	*0.3*
4	*6.0*	*6.0*	*6.0*	*6.0*	*0.4*
5	*3.0*	*7.0*	*3.0*	*7.0*	*0.5*

Two initial points $x = (1,1,1,1)$ and $(6,6,6,6)$ are used. The proposed filled function approach succeeds in identifying the global minimum solution:

$$x^*=(4.00003715282, 4.00013327659, 4.00003715282, 4.00013327659)$$

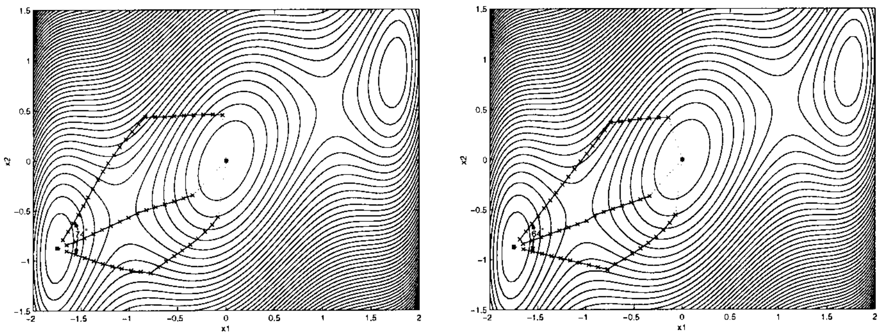
and $f(x^*)=-10.15319967906$, with $m=80$ and $\delta=0.1$. The computational results are summarized in Tables XII and XIII, respectively.

Problem 8 (n-dimensional function in [5]).

$$\min f(x)=\frac{\pi}{n}[10\sin^2\pi x_1+g(x)+(x_n-1)^2],$$
$$\text{s.t. } -10\leqslant x_i\leqslant 10,\quad i=1,2,\ldots,n,$$

where

$$g(x)=\sum_{i=1}^{n-1}[(x_i-1)^2(1+10\sin^2\pi x_{i+1})].$$



(a) $\rho = 1$, $\mu = 0.1$, Range of success $= [-18°, 56°]$

(b) $\rho = 1$, $\mu = 0$, Range of success $= [-13°, 51°]$

Figure 2. Contours of $f(x)$ and the ranges of success in Problem 2.

Five sizes of the problem are considered in the test, $n = 2, 3, 5, 7, 10$. The proposed filled function approach succeeds in identifying the global minimum solutions: $x^* = (1, 1, \ldots, 1)$ and $f(x^*) = 0$ for all n, with $m = 2n$ and $\delta = 0.1$. The computational results are summarized in Tables XIV–XVIII, respectively.

5. Conclusions

A new version of a filled function is proposed in this paper with much improved performance in finding a global minimum solution. Compared to the local search where the local information, such as the gradient and Hessian, can be used to determine a search direction, there is always a lack of global information in determining a search direction in global optimization. In our algorithm, a better minimum is sought starting from a set of initial points that are symmetry to the current minimum point. Selection of the parameters ρ and μ affects the size of the range starting from which a search will succeed in locating another better local minimum. For example, in Problem 2 (see Figure 2), if ρ and μ are assigned to 1 and 0.1, respectively, the range of success is from $-18°$ to $56°$ (a total of $74°$). On the other hand, if ρ and μ are assigned to 1 and 0, respectively, the range of success is from $-13°$ to $51°$ (a total of $64°$) only. Although the successful rate in finding another better minimum will be increased by increasing the density of the initial points, this growth in number of initial points, unfortunately, is exponential with respect to the dimension of the problem. Indeed, if the number of grid points in each plane is $4k, k \geqslant 1$, the total number of initial points is at least $2n$, for $k = 1$, and $[(2k-1)^n - 1]/(k-1)$, for $k > 1$. For example, suppose that the problem dimension is $n = 3$ and the number of grid points in each plane is 4 ($k = 1$), then the minimum number of initial points is 6 only. However, if the number of grid points in each plane is 8 ($k = 2$), then the total number of initial points becomes at least 26. For large-scale problems, some global structural information of the problem must be used to reduce this curse of dimensionality and to facilitate the identification of a global search direction.

6. Acknowledgements

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Table I. Computational results for problem 1 with $c=0.2$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(6, -2)	–	–	(5.7221, -1.8806)	2.5070
2	$x_1^*-(0.1,0)$	1	1e-3	(4.7387, -1.7417)	1.6212
3	$x_2^*+(0,0.1)$	1	1e-3	(4.7096, -1.3985)	1.3566
4	$x_3^*-(0.1,0)$	1	1e-3	(3.7387, -1.2649)	0.61647
5	$x_4^*-(0.1,0)$	1	1e-3	(2.7380, -0.78836)	0.088673
6	$x_5^*-(0.1,0)$	1	1e-3	(1.8784, -0.34585)	0

Table II. Computational results for Problem 1 with $c=0.5$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(0, 0)	–	–	(0.042023, -0.094772)	0.51745
2	$x_1^*+(0.1,0)$	1	1	(0.99991, -1.2524e-4)	2.2389e-7
3	$x_2^*+(0,0.1)$	1	1e-1	(1.0000, -2.2205e-14)	0

Table III. Computational results for Problem 1 with $c=0.05$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(10, -10)	–	–	(8.7299, -3.2965)	9.0733
2	$x_1^*-(0.1,0)$	1	1e-3	(7.7280, -2.8347)	6.5031
3	$x_2^*-(0.1,0)$	1	1e-3	(6.7248, -2.3724)	4.3943
4	$x_3^*-(0.1,0)$	1	1e-3	(5.7198, -1.9162)	2.7434
5	$x_4^*-(0.1,0)$	1	1e-3	(4.7129, -1.4891)	1.5351
6	$x_5^*-(0.1,0)$	1	1e-3	(3.7305, -1.2306)	0.61844
7	$x_6^*-(0.1,0)$	1	1e-3	(2.7300, -0.79341)	0.10216
8	$x_7^*-(0.1,0)$	1	1e-3	(1.8513, -0.40209)	0

Table IV. Computational results for Problem 2 with initial point $(-2,-1)$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(-2, -1)	–	–	(-1.7476, -0.87378)	0.29864
2	$x_1^*+(0.1,0)$	1	1e-1	(0, 0)	0

Table V. Computational results for Problem 2 with initial point $(2,1)$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(2, 1)	–	–	(1.7476, 0.87378)	0.29864
2	$x_1^* - (0.1,0)$	1	1e-4	(0, 0)	0

Table VI. Computational results for Problem 3 with initial point $(-2,1)$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(-2, 1)	–	–	(-1.6071, 0.56865)	2.1043
2	$x_1^* + (0.1,0)$	1	1	(0.089842, 0.71266)	-1.0316

Table VII. Computational results for problem 3 with initial point $(2,-1)$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(2, -1)	–	–	(1.6071, -0.56865)	2.1043
2	$x_1^* - (0.1,0)$	1	1e-5	(-0.089842, -0.71266)	-1.0316

Table VIII. Computational results for Problem 3 with initial point $(-2,-1)$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(-2, -1)	–	–	(-1.7036, -0.79608)	-0.21546
2	$x_1^* + (0.1,0)$	1	1e-1	(-0.089842, -0.71266)	-1.0316

Table IX. Computational results for Problem 4

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(-1, 0)	–	–	(-1.0000, 0)	1.0000
2	$x_1^* + (0.1,0)$	1	1	(0, 0)	0

Table X. Computational results for Problem 5

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(-1, -1)	–	–	(-0.60000, -0.40000)	30.000
2	$x_1^* + (0.1,0)$	1	1e-5	(0, -1.0000)	3.0000

Table XI. Computational results for Problem 6

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(1, 1)	–	–	(1.0865, 1.0865)	2.8841e-17
2	$x_1^*+(0.1,0)$	1	1	(1.3200, 1.8703e-12)	-13.052
3	$x_2^*+(0,0.1)$	1	1e-5	(1.3200, 4.8581)	-37.681
4	$x_3^*+(0.1,0)$	1	1e-5	(3.2800, 4.8581)	-46.511
5	$x_4^*+(0.1,0)$	1	1e-5	(4.2760, 4.8581)	-79.411
6	$x_5^*+(0.1,0)$	1	1e-6	(5.4829, 4.8581)	-186.73

Table XII. Computational results for Problem 7 with initial point $(1,1,1,1)$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(1, 1, 1, 1)	–	–	(1.0001, 1.0002, 1.0001, 1.0002)	-5.0552
2	$x_1^*+0.1(\cos(\frac{\pi}{4}), \sin(\frac{\pi}{4})\cos(\frac{\pi}{4}), \sin^2(\frac{\pi}{4})\cos(\frac{\pi}{4}), \sin^3(\frac{\pi}{4}))$	1	1e-2	(4.0000, 4.0001, 4.0000, 4.0001)	-10.153

Table XIII. Computational results for Problem 7 with initial point $(6,6,6,6)$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(6, 6, 6, 6)	–	–	(5.9987, 6.0003, 5.9987, 6.0003)	-2.6829
2	$x_1^*+0.1(\cos(\frac{\pi}{4}), \sin(\frac{\pi}{4})\cos(\frac{\pi}{4}), \sin^2(\frac{\pi}{4})\cos(\frac{\pi}{4}), \sin^3(\frac{\pi}{4}))$	1	1e-1	(7.9996, 7.9996, 7.9996, 7.9996)	-5.1008
3	$x_2^*+0.1(\cos(\frac{3\pi}{4}), \sin(\frac{3\pi}{4})\cos(\frac{3\pi}{4}), \sin^2(\frac{3\pi}{4})\cos(\frac{5\pi}{4}), \sin^2(\frac{3\pi}{4})\sin(\frac{5\pi}{4}))$	1	1e-2	(4.0000, 4.0001, 4.0000, 4.0001)	-10.153

Table XIV. Computational results for Problem 8 with $n=2$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(-4, -4)	–	–	(-3.9490, -3.9979)	78.126
2	$x_1^*+(0.1,0)$	1	1e-3	(-2.9594, -3.9968)	64.124
3	$x_2^*+(0.1,0)$	1	1e-3	(-0.97956, -3.9871)	45.389
4	$x_3^*+(0.1,0)$	1	1e-4	(0.012700, -3.9476)	40.418
5	$x_4^*+(0.1,0)$	1	1e-4	(1.0000, 1.0000)	0

Table XV. Computational results for Problem 8 with $n=3$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(-3, -3, -3)	–	–	(-2.9594, -2.9974, -2.9975)	50.075
2	$x_1^*+(0.1,0,0)$	1	1e-3	(-1.9697, -2.9954, -2.9975)	42.810
3	$x_2^*+(0.1,0,0)$	1	1e-3	(-0.97967, -2.9897, -2.9975)	37.603
4	$x_3^*+(0.1,0,0)$	1	1e-3	(0.011723, -2.9584, -2.9974)	34.363
5	$x_4^*+(0.1,0,0)$	1	1e-3	(1.0000, 1.0000, 1.0000)	0

Table XVI. Computational results for Problem 8 with $n=5$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(-1, -1, -1, -1, -1)	–	–	(-0.97983, -0.99483, -0.99491, -0.99491, -0.99492)	12.515
2	$x_1^*+(0.1,0,0,0,0)$	1	1e-3	(0.010455, -0.97941, -0.99483, -0.99491, -0.99492)	10.630
3	$x_2^*+(0.1,0,0,0,0)$	1	1e-3	(1.0000, 1.0000, 1.0000, 1.0000, 1.0000)	0

Table XVII. Computational results for Problem 8 with $n = 7$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(2, 2, 2, 2, 2, 2, 2)	–	–	(1.9899, 1.9897, 1.9896, 1.9896, 1.9896, 1.9896, 1.9898)	3.1095
2	$x_1^* -$ $(0.1, 0, 0, 0, 0, 0, 0)$	1	1e-3	(1.0000, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000)	0

Table XVIII. Computational results for Problem 8 with $n = 10$

k	$x_k^{(0)}$	ρ	μ	x_k^*	$f(x_k^*)$
1	(6, 6, 6, 6, 6, 6, 6, 6, 6, 6)	–	–	(5.9490, 5.9979, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980)	78.432
2	x_1^*- (0.1, 0, 0, 0, 0, 0, 0, 0, 0, 0)	1	1e-3	(-1.9696, 5.9943, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980)	73.450
3	x_2^*+ (0.1, 0, 0, 0, 0, 0, 0, 0, 0, 0)	1	1e-3	(-0.97956, 5.9871, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980)	71.884
4	x_3^*+ (0.1, 0, 0, 0, 0, 0, 0, 0, 0, 0)	1	1e-3	(0.012709, 5.9476, 5.9979, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980, 5.9980)	70.890
5	x_4^*+ (0.1, 0, 0, 0, 0, 0, 0, 0, 0, 0)	1	1e-3	(1.0000, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000)	0